Numerical Simulation of Fluid Flow Based on the Unified Coordinates

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Most numerical simulations of fluid flow are based on the Eulerian system of coordinates. It is relatively simple, but

(a) it smears contact discontinuities badly;

(b) it needs generating a body-fitted mesh for flow past a body.

Need to explore other systems, and we propose a unified coordinate system.
Content

Introduction

The Unified Coordinates (UC)

1-D Flow

2-D and 3-D Flow in UC (including Lagrangian Coordinates)

Movement of the UC

Automatic Mesh Generation

Conclusions
Introduction

Two coordinate systems for describing fluid motion have existed:

* Eulerian coordinates (mesh) are fixed in space
* Lagrangian coordinates (mesh) move with fluid

Are they equivalent theoretically?

“Yes” for 1-D flow (D H Wagner, *J. Diff. Eq.*, 64, 118-136, 1987)

Computationally, they are not equivalent, even for 1-D flow.

So, an important issue arises:

To search for an “optimal” coordinate system
“Optimal” Coordinates

For compressible flow, we want a coordinate system to have the following properties:

(a) Conservation PDEs exist, as in Eulerian;
(b) Contacts are sharply resolved, as in Lagrangian;
(c) Body-fitted mesh can be generated automatically;
(d) Mesh to be orthogonal;
(e) Mesh to be uniform.

The unified coordinate system satisfies these requirements
The Unified Coordinates \((\lambda, \xi, \eta, \zeta)\) are given by the transformation

\[
\begin{align*}
  dt &= d\lambda \\
  dx &= Ud\lambda + Ad\xi + Ld\eta + Pd\zeta \\
  dy &= Vd\lambda + Bd\xi + Md\eta + Qd\zeta \\
  dz &= Wd\lambda + Cd\xi + Nd\eta + Rd\zeta
\end{align*}
\]

We get

\[
\frac{D\xi}{Dt} \left( \begin{array}{c}
  \xi \\
  \eta \\
  \zeta
\end{array} \right) = 0 \quad \frac{D\phi}{Dt} \equiv \frac{\partial}{\partial t} + \tilde{Q} \cdot \nabla_x
\]

So \((\xi, \eta, \zeta)\), and hence the computational cells, move with the pseudo particle whose velocity is \(\tilde{Q} = (U, V, W)\) which is arbitrary
Compatibilty Conditions

\[
\begin{align*}
\frac{\partial A}{\partial \lambda} &= \frac{\partial U}{\partial \xi}, & \frac{\partial L}{\partial \lambda} &= \frac{\partial U}{\partial \eta}, & \frac{\partial P}{\partial \lambda} &= \frac{\partial U}{\partial \varsigma}, \\
\frac{\partial B}{\partial \lambda} &= \frac{\partial V}{\partial \xi}, & \frac{\partial M}{\partial \lambda} &= \frac{\partial V}{\partial \eta}, & \frac{\partial Q}{\partial \lambda} &= \frac{\partial V}{\partial \varsigma}, \\
\frac{\partial C}{\partial \lambda} &= \frac{\partial W}{\partial \xi}, & \frac{\partial N}{\partial \lambda} &= \frac{\partial W}{\partial \eta}, & \frac{\partial R}{\partial \lambda} &= \frac{\partial W}{\partial \varsigma}.
\end{align*}
\]

Time evolution

\[
\begin{align*}
\frac{\partial A}{\partial \eta} &= \frac{\partial L}{\partial \xi}, & \frac{\partial A}{\partial \varsigma} &= \frac{\partial P}{\partial \xi}, & \frac{\partial L}{\partial \varsigma} &= \frac{\partial P}{\partial \eta}, \\
\frac{\partial B}{\partial \eta} &= \frac{\partial M}{\partial \xi}, & \frac{\partial B}{\partial \varsigma} &= \frac{\partial Q}{\partial \xi}, & \frac{\partial M}{\partial \varsigma} &= \frac{\partial Q}{\partial \eta}, \\
\frac{\partial C}{\partial \eta} &= \frac{\partial N}{\partial \xi}, & \frac{\partial C}{\partial \varsigma} &= \frac{\partial R}{\partial \xi}, & \frac{\partial N}{\partial \varsigma} &= \frac{\partial R}{\partial \eta}.
\end{align*}
\]

Differential constraints
Special Cases:

Eulerian \( Q = 0 \)
Lagrangian \( Q = q, \) \( q \) being fluid velocity

General case, we have a unified (Euler-Lagrangian) coordinate system with 3 degrees of freedom: \( U, V \) and \( W \) are arbitrary.

Need 3 conditions to determine the mesh velocity

\[ Q = (U, V, W) \]
One-Dimensional Flow

It can be shown that

*UC*, *ie*, Lagrangian coordinate + shock-adaptive Godunov scheme

*is superior to Eulerian system*
Ex 1. Contact Resolution in A Riemann Problem (Godunov-MUSCL)

UC (Lagrangian + Adaptive-Godunov scheme)
2-D & 3-D flow

Eulerian method is relatively **simple**, because the Euler equations of gas dynamics can be written in **conservation PDE form**, which is the basis for shock-capturing methods, but

(a) it smears contact interfaces badly, and

(b) it needs generating a body-fitted mesh.

Lagrangian method **resolves contact and material interfaces sharply**, because they **coincide with the coordinate surfaces**, but

(a) it breaks down due to cell deformation,

(b) the gas dynamics equations could not be written in conservation PDE form,

(c) it also needs mesh generation.
2-D Gas Dynamic Equations

In Eulerian Coordinates

\[
\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \quad (E)
\]

where

\[
E = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho u \left( e + \frac{p}{\rho} \right) \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho v \left( e + \frac{p}{\rho} \right) \end{pmatrix}
\]

\[
e = \frac{1}{2} (u^2 + v^2) + \frac{1}{\gamma - 1} \frac{p}{\rho}
\]

Eq. (E) is: (a) in conservation form, (b) hyperbolic in \( t \)
Transforming to the unified coordinates, (E) become

\[
\frac{\partial E}{\partial \lambda} + \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} = 0 \tag{U}
\]

where

\[
E = \begin{pmatrix} \rho J \\ \rho J u \\ \rho J v \\ \rho J e \\ A \\ B \\ L \\ M \end{pmatrix}, \quad F = \begin{pmatrix} \rho X \\ \rho X u + p M \\ \rho X v - p L \\ \rho X e + p(u M - v L) \\ -U \\ -V \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} \rho Y \\ \rho Y u - p B \\ \rho Y v + p A \\ \rho Y e + p(v A - u B) \\ 0 \\ 0 \\ -U \\ -V \end{pmatrix}
\]

With \( J = AM - BL \), \( X = (u - U)M - (v - V)L \) and \( Y = (v - V)A - (u - U)B \).

Eq (U) is: (a) a closed system in conservation form, (b) hyperbolic in \( \lambda \), except Lagrangian when \( U = u \) and \( V = v \).
Remarks

There are two important consequences of having a closed system of governing equations in conservation form:

(a) Effects of moving mesh on the flow are fully accounted for;

(b) The system can be solved as easily as the Eulerian one.
Determination of Mesh Velocity \( (U, V) \)

We place two requirements:

(A) coordinate lines \( \eta = \text{const.} \) shall be material lines of fluid particles, meaning \( \frac{D\eta}{Dt} = 0 \). With this gives

\[
(V - v) A - (U - u) B = 0
\]  

Observations:

(1) Contact lines, being material lines, coincide with coordinate lines and, therefore, can be resolved sharply.

(2) As the body surface is a material line, condition (a) guarantees that the unified mesh is automatically a body-fitted mesh.

(3) Material surfaces (including free surfaces), correspond to \( \eta = \text{const} \)

(4) \( \eta (x, y, t) \) is a level set function.
(B) Mesh angles and orthogonality shall be preserved, yielding an ODE for $U$

$$\frac{\partial U}{\partial \eta} + P(\eta; \lambda, \xi) U = Q(\eta; \lambda, \xi)$$

$U (\eta)$ prescribed at $\eta = \text{const.}$

$$P(\eta, \lambda, \xi) = \frac{S^2}{T^2 J} \left( A \frac{\partial B}{\partial \xi} - B \frac{\partial A}{\partial \xi} \right) - \frac{L}{A J} \left( A \frac{\partial B}{\partial \eta} - B \frac{\partial A}{\partial \eta} \right)$$

$$Q(\eta, \lambda, \xi) = \frac{S^2 A}{T^2 J} \left( B \frac{\partial u}{\partial \xi} - A \frac{\partial v}{\partial \xi} \right) + \frac{L}{J} \left( A \frac{\partial v}{\partial \eta} - B \frac{\partial u}{\partial \eta} \right) + u P(\eta, \lambda, \xi)$$

$$S^2 = L^2 + M^2, \quad T^2 = A^2 + B^2$$

Alternatively, we may require the Jacobian be preserved.
**UC Computation**

As we have conservation form, computations are done like Eulerian in $\lambda$-$\xi$-$\eta$ space by marching in time $\lambda$, at each time step the mesh velocity ($U, V$) is computed (Godunov-MUSCL scheme plus splitting).
Special Case: Lagrangian Gas Dynamics

For $U = u$ and $V = v$, we get

$$\frac{\partial E}{\partial \lambda} + \frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \eta} = 0 \quad (L)$$

where

$$E = \begin{pmatrix} \rho J \\ \rho Ju \\ \rho Jv \\ \rho Je \\ A \\ B \\ L \\ M \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ pM \\ -pL \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ -pB \\ pA \\ p(vA - uB) \end{pmatrix}$$
Remarks

(1) The gas dynamics equations in Lagrangian coordinates are written in conservation form for the first time, rendering Lagrangian scheme a moving mesh scheme in Eulerian space.

(2) Lagrangian GD equation is only weakly hyperbolic: all eigenvalues are real, but there is no complete set of linearly independent eigenvectors. This is also true for 3-D case. Hence, Lagrangian GD is not equivalent to Eulerian GD.

(3) Despres & Mazeran (2005) argue that Lagrangian reformulation of Eulerian hyperbolic system will lead to weakly hyperbolic system. (1-D is an exception)
Automatic Mesh Generation
Example 2: Steady supersonic airfoil
(Space-marching method)
\[ \lambda = 0.0 \]
$\lambda = 0.1$
\[ \lambda = 0.3 \]
$\lambda = 0.7$
Surface Mach number distribution, 120 cells
Computing time: 1.8s (P4, 2.8GHz)
Surface Mach Number Distributions on Diamond-Shaped Airfoil.
Eulerian Computation (5th Order WENO Scheme), 100 x 200 cells.
Computing Time to 20,000 Steps: 2,393s on P4, 2.8 GHz.
Example 3. $M = 0.8$ past a NACA 0012 airfoil
(WH Hui & JJ Hu)
Pressure at $t=10.1$

Hafez et al. [4]
Present

Hafez et al. [4]
Example 4
Two-Fluids Flow past a NACA 0012 airfoil
$M = 2.2$, $AoA = 8$ degrees (Hui & Shyue)
Example 5: Supersonic flow past a pitching oscillating diamond-shape airfoil

( G P Zhao & W H Hui)

$M_\infty = 3$

$\theta = \theta_0 \sin \omega t$
Unsteady pressure distribution

\[ M_\alpha = 3 \]

\[ \theta = 2^\circ \sin 30 t \]

\[ \alpha = 10^\circ \]
Example 6

Free falling leaves

by

Changqiu Jin and Kun Xu

HKUST
falling leaves with fluttering and tumbling motion
Computed paths (10 full rotations)

fluttering
tumbling
Mesh, once generated, is fixed and moved with the body; this gives mesh velocity \((U, V)\) at all time.
Computational procedure:

(1) At time $t$, the body position and mesh velocity are given;

(2) Use Xu’s gas kinetic-BGK solver to find the flow and hence the aerodynamic forces on the body (Note: the effects of mesh movement on the flow are correctly and fully accounted for through the use of geometric conservation laws in UC);

(3) Use Newton’s 2nd law to compute the motion of the body under the aerodynamic and gravitational forces, yielding the body position and the mesh movement at new time;

Repeat (1) – (3).
measured by experiment

Computed by HKUST

Computed by Cornell group
Conclusions

Good numerical simulation of fluid flow requires two ingredients: an *optimally moving coordinate system* and an *accurate flow solver*.

The *unified coordinate system*, being flexible and having closed conservation form PDE, provides *an ideal framework* to take advantages of many well-developed flow solvers.
Thank You

(Ref: W.H. Hui, The unified coordinate system in computational fluid dynamics, Communications in Computational Physics, 2, 577-610, 2007)
Definition of Unified Coordinates

1-D \[ \frac{D_q \xi}{Dt} = 0 \] thus \( U = u \), but \( A \) is arbitrary
(classical Lagrangian: \( U = u \), \( A = 1/\rho \))
plus adaptive Godunov scheme

2-D \[ \frac{D_q \xi}{Dt} = 0 \] plus mesh-angle preserving
(or Jacobian preserving)

3-D \[ \frac{D_q \xi}{Dt} = 0 \] \( \frac{D_q \eta}{Dt} = 0 \)
plus mesh-skewness preserving
(or Jacobian preserving)

\( UC \) combines the advantages of Lagrangian
and Eulerian coordinates, and beyond
By transformation
\[
\begin{cases}
    dt = d\lambda \\
    dx = ud\lambda + \frac{1}{\rho}d\xi
\end{cases}
\]
we get
\[
\frac{\partial}{\partial \lambda} \begin{pmatrix}
\frac{1}{\rho} \\
u \\
e
\end{pmatrix} + \frac{\partial}{\partial \xi} \begin{pmatrix}
-u \\
p \\
up
\end{pmatrix} = 0
\]
(Classical Lagrangian, conservation form and hyperbolic)
Using UC, all defects of Eulerian shock-capturing methods, namely

- contact smearing
- wall-overheating
- start-up errors
- slow moving shocks
- low-density flow
- sonic-point glitch
- strong rarefaction wave

are cured or avoided.
Special Case: Steady Flow

In the special case of steady flow

\[ \frac{Dq\eta}{Dt} = 0 \quad \text{becomes} \quad Q \ast \nabla \eta = 0 \]

\[ \frac{Dq\eta}{Dt} = 0 \quad \text{becomes} \quad q \ast \nabla \eta = 0 \]

Hence \( Q = h \ q \)

For steady supersonic flow, space-marching method is most effective.
“As I am not pressed and work more for my pleasure than from duty, ...........
I make, unmake, and remake, until I am passably satisfied with my results, which happens only rarely”

Joseph-Louis Lagrange (1736-1813)